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Asymptotic Properties of Powers of Kantorovič Operators

JOSEF NAGEL

Karl-Maybach-Gymnasium, Wendelgardstrasse, D-7990 Friedrichshafen 1, West Germany

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1. INTRODUCTION

In a previous publication [8] we have studied the limit behaviour of sequences of powers of Bernstein operators. It seems natural to ask for which of the other approximation operators one can establish similar convergence theorems and Voronovskaja type theorems. In this note we deal with the Kantorovič operators P_n ($n \in \mathbb{N}_0$) defined by

$$P_n(f; x) = (n+1) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{k/n+1}^{k+1/n+1} f(t)\, dt \qquad (x \in [0, 1])$$

for functions $f \in \mathscr{L}_1[0, 1]$ (Kantorovič [2]; Lorentz [4, p. 30]).

The Kantorovič operators P_n are closely related to the Bernstein operators B_n; more exactly,

$$P_n(f; x) = \frac{d}{dx} B_{n+1}(F; x) \tag{1}$$

for each $n \in \mathbb{N}_0$, with $F(x) = \int_0^x f(t)\, dt$ (cf. [4, p. 30]). By induction, this relation can be extended to integral powers. One gets

$$P_n^k(f; x) = \frac{d}{dx} B_{n+1}^k(F; x) \tag{2}$$

for all $n \in \mathbb{N}_0$, $k \in \mathbb{N}$. This relation makes it possible to establish for each theorem in [8] the parallel theorem for the Kantorovič operators.

In the following, let $(k_n)_{n \in \mathbb{N}}$ always be a sequence of natural numbers. In Section 2 we shall determine $\lim_{n \to \infty} P_n^{k_n} f$ for functions $f \in \mathscr{L}_2[0, 1]$, provided $\lim_{n \to \infty}(k_n/n)$ exists. This section is essentially based on our earlier work [8]. As a secondary result, we shall obtain that the limits of the eigenfunctions of

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the Kantorovič operators are the Legendre polynomials on the interval $[0, 1]$. In Section 3 we shall state theorems of Voronovskaja type for the marginal cases $\lim_{n\to\infty}(k_n/n)=0$ and $\lim_{n\to\infty}(k_n/n)=\infty$.

We shall use the following abbreviations: For each $s \in \mathbb{N}_0$, $\mathscr{P}_s$ is the space of all real polynomials with maximal degree s, and e_s is the monomial with $e_s(x)=x^s$ $(x \in [0, 1])$. For each function $f \in \mathscr{L}_1[0, 1]$, F will denote the integral function with $F(x)=\int_0^x f(t)\,dt$. For all unexplained notation we refer to [8].

2. Convergence Theorems

We begin at the asymptotic behaviour of $(P_n^{k_n}p)_{n\in\mathbb{N}}$ for polynomials p, and assume p is a given polynomial of degree s $(s \in \mathbb{N})$. In the sequel, we shall only consider indices n with $n \geqslant s$.

For each n, the polynomial space $\mathscr{P}_s$ is an invariant subspace of P_n. $\mathscr{P}_s$ is spanned by the eigenfunctions $p'_{j+1,n+1}$ of P_n, which are polynomials of degree j and the derivatives of the eigenfunctions $p_{j+1,n+1}$ of the Bernstein operator B_{n+1} $(j=0, 1, 2,..., s)$; the eigenvalues of P_n are 1 and $(1-1/(n+1))(1-2/(n+1)) \cdots (1-j/(n+1))$ $(j=1, 2,..., s)$. Hence there holds $P_n p'_{1,n+1}=p'_{1,n+1}$ and $P_n p'_{j+1,n+1}=(1-1/(n+1))(1-2/(n+1)) \cdots (1-j/(n+1))\,p'_{j+1,n+1}$ $(j=1, 2,..., s)$. These facts ensue from the corresponding facts for the Bernstein operators and on (1). For further details we refer to [8]. Kelisky and Rivlin [3] have shown that the polynomials $p_{j+1,n+1}$ are coefficientwise convergent (as $n\to\infty$), and they calculated the limit polynomials p_{j+1}. As in [8], in the following we will use the denotation $\lim_{n\to\infty} p_{j+1,n+1} \stackrel{c}{=} p_{j+1}$ for *coefficientwise convergence.*

Multiplying with the factor $(2j+1)^{-1/2}\binom{2j+1}{j+1}$, we get the polynomials $g_{j,n}=(2j+1)^{-1/2}\binom{2j+1}{j+1}p'_{j+1,n+1}$ and $g_j=(2j+1)^{-1/2}\binom{2j+1}{j+1}p'_{j+1}$. Then there holds $\lim_{n\to\infty} g_{j,n} \stackrel{c}{=} g_j$ $(j=0, 1, 2,..., s)$, and g_j is the Legendre polynomial of degree j on the interval $[0, 1]$ as we have shown in [8]. For $j=0$, from $p_{1,n}=p_1=e_1$ [8, Lemma 1], follows $g_{0,n}=g_0=e_0$, and for $j=1$, $g_1$is the polynomial with $g_1(x)=\sqrt{3}(2x-1)$. For powers P_n^k we obtain

$$P_n^k g_{0,n}=g_{0,n}$$

and

$$P_n^k g_{j,n}=\left(1-\frac{1}{n+1}\right)^k\left(1-\frac{2}{n+1}\right)^k\cdots\left(1-\frac{j}{n+1}\right)^k g_{j,n} \qquad (j=1, 2,..., s). \tag{3}$$

For each $n \geqslant s$, the given polynomial p can be represented in the form

$$p = \sum_{j=0}^{s} a_{j,n} g_{j,n} \tag{4}$$

with unique coefficients $a_{j,n}$; similarly, there exists a unique representation

$$p = \sum_{j=0}^{s} a_j g_j, \tag{5}$$

where a_j are the coefficients of the Legendre expansion of p. In particular, there holds $a_0 = \int_0^1 p(t)\,dt$ and $a_1 = \sqrt{3} \int_0^1 t(1-t)\,p'(t)\,dt$. Since $p_{j+1,n+1}(0) = p_{j+1,n+1}(1) = 0$ (for $j = 1, 2, \ldots, s$) and $p_{1,n}(x) = x$ (cf. [8, Lemma 1]), integrating (4) we also find $a_{0,n} = \int_0^1 p(t)\,dt$. For the other coefficients $a_{j,n}$, it is readily proved by induction (cf. [8, Lemma 1]) that $\lim_{n \to \infty} a_{j,n} = a_j$ for each $j \in \{1, 2, \ldots, s\}$. Combining all these facts and using Lemma 2 from [8], we obtain the following proposition.

PROPOSITION 1. *Let p be the given polynomial and $(k_n)_{n \in \mathbb{N}}$ a sequence of natural numbers.*

(i) *In the case* $\lim_{n \to \infty}(k_n/n) = 0$ *there holds*

$$\lim_{n \to \infty} P_n^{k_n} p \overset{c}{=} a_0 g_0 + a_1 g_1 + \cdots + a_s g_s = p. \tag{6}$$

As to the degree of approximation there holds

$$\lim_{n \to \infty} \frac{n}{k_n} \{P_n^{k_n} p - p\} \overset{c}{=} \lim_{n \to \infty} \frac{n}{k_n} \sum_{j=1}^{s} a_{j,n} \{P_n^{k_n} g_{j,n} - g_{j,n}\}$$

$$\overset{c}{=} -\sum_{j=1}^{s} \binom{j+1}{2} a_j g_j. \tag{7}$$

For the limit function in (7) *there holds*

$$-\sum_{j=1}^{s} \binom{j+1}{2} a_j g_j(x) = \left(\frac{1}{2} x(1-x)\,p'(x)\right)'.$$

This follows from $-\binom{j+1}{2} g_j(x) = (\frac{1}{2} x(1-x)\,g_j'(x))'$ *for* $j = 1, 2, \ldots, s$, *which is proved as* (5′) *in* [8].

(ii) *In the case* $\lim_{n \to \infty}(k_n/n) = \infty$ *there holds*

$$\lim_{n \to \infty} P_n^{k_n} p \overset{c}{=} a_0 g_0, \tag{8}$$

with $a_0 g_0(x) = \int_0^1 p(t)\,dt$. *As to the degree of approximation there holds*

$$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{-k_n} \left\{P_n^{k_n} p - \int_0^1 p(t)\,dt e_0\right\} \stackrel{c}{=} a_1 g_1, \tag{9}$$

with

$$a_1 g_1(x) = (x - \tfrac{1}{2}) \int_0^1 6t(1-t)\,p'(t)\,dt.$$

(iii) *In the case* $\lim_{n\to\infty}(k_n/n) = q$ *with* $q \in (0, \infty)$ *there holds*

$$\lim_{n\to\infty} P_n^{k_n} p \stackrel{c}{=} a_0 g_0 + E_1^q a_1 g_1 + E_2^q a_2 g_2 + \cdots + E_s^q a_s g_s, \tag{10}$$

where $E_j = e^{-j(j+1)/2}$. *In this case a simple result concerning the degree of approximation seems to be impossible.*

It will be our next aim to extend the limit relations (6), (8), and (10) to wider classes of functions. Relations (7) and (9) concerning the degree of approximation will be extended in Section 3.

For Theorem 1, we have chosen a $\mathscr{L}_2$-version, which seems to be the most natural. Obviously similar uniform or $\mathscr{L}_p$-versions for the existence of the limits are readily available. But for the representation of the limit operators cf. the remarks in [8] after Theorem 1.

In the following theorem all occurring limit relations are to be understood with respect to the $\mathscr{L}_2$-norm. $\langle \cdot, \cdot \rangle$ denotes the inner product in the Hilbert space $\mathscr{L}_2[0, 1]$. We again use the abbreviation $E_j = e^{-j(j+1)/2}$. For $q = \infty$, we set $E_j^q = 1$ if $j = 0$ and $E_j^q = 0$ otherwise.

THEOREM 1. (i) *For each* $q \in [0, \infty]$,

$$\mathfrak{C}_q f = \sum_{j=0}^{\infty} E_j^q \langle f, g_j \rangle g_j \qquad (f \in \mathscr{L}_2[0, 1])$$

is a linear bounded operator from $\mathscr{L}_2[0, 1]$ *into itself with operator norm* $\|\mathfrak{C}_q\|_2 = 1$. *In particular,* $\mathfrak{C}_0$ *is the identity in* $\mathscr{L}_2[0, 1]$, *and* $\mathfrak{C}_\infty$ *is the operator with*

$$\mathfrak{C}_\infty(f; x) = \int_0^1 f(t)\,dt.$$

(ii) *If* $(k_n)_{n\in\mathbb{N}}$ *is a sequence of natural numbers with* $\lim_{n\to\infty}(k_n/n) = q$, *then there holds* $\lim_{n\to\infty} P_n^{k_n} f = \mathfrak{C}_q f$ *for each* $f \in \mathscr{L}_2[0, 1]$.

(iii) $\{\mathfrak{C}_q \mid 0 \leqslant q < \infty\}$ *is a strongly continuous semigroup of linear bounded operators on* $\mathscr{L}_2[0, 1]$.

Proof. It is known (cf. [4, p. 32]) that $\|P_n\|_2 = 1$ for all $n \in \mathbb{N}_0$. Hence (i) and (ii) follow from Proposition 1 and the theorem of Banach and Steinhaus. The details of the proof, which is straightforward, are left to the reader. ∎

3. Voronovskaja Type Theorems for the Cases $\lim_{n\to\infty} (k_n/n) = 0$ and $\lim_{n\to\infty} (k_n/n) = \infty$

We first consider the case $\lim_{n\to\infty}(k_n/n) = 0$. In this case the approximation property $\lim_{n\to\infty} P_n^{k_n} f = f$ *uniformly on* $[0, 1]$ *for each* $f \in \mathscr{C}[0, 1]$ also ensues from the theorem of Bohman and Korovkin. To estimate the quality of this approximation, we need the defects of approximation with the functions of the test set $\{e_0, e_1, e_2\}$, namely,

$$d_{0,n}(x) = P_n^{k_n}(e_0; x) - e_0(x) = 0,$$

$$d_{1,n}(x) = P_n^{k_n}(e_1; x) - e_1(x) = \left(\left(1 - \frac{1}{n+1}\right)^{k_n} - 1\right)\left(x - \frac{1}{2}\right),$$

$$d_{2,n}(x) = P_n^{k_n}(e_2; x) - e_2(x) = \left(1 - \frac{1}{n+1}\right)^{k_n}\left(1 - \frac{2}{n+1}\right)^{k_n}\left(x^2 - x + \frac{1}{6}\right) + \left(1 - \frac{1}{n+1}\right)^{k_n}\left(x - \frac{1}{2}\right) - x^2 + \frac{1}{3}. \tag{11}$$

Employing Theorem 6.1 from [7], we obtain

$$\begin{aligned}
&\left|\frac{n}{k_n}\left\{P_n^{k_n}(f; x) - f(x)\right\} - \frac{n}{k_n}\left\{\left(1 - \frac{1}{n+1}\right)^{k_n} - 1\right\}\left(x - \frac{1}{2}\right) f'(x)\right| \\
&\quad \leqslant \left[\frac{n}{k_n}\left\{1 - \left(1 - \frac{1}{n+1}\right)^{k_n}\left(1 - \frac{2}{n+1}\right)^{k_n}\right\}\right. \\
&\quad\quad \left. - 2\frac{n}{k_n}\left\{1 - \left(1 - \frac{1}{n+1}\right)^{k_n}\right\}\right]\frac{M}{2}(x - x^2) \\
&\quad\quad + \frac{M}{12}\left[\frac{n}{k_n}\left\{\left(1 - \frac{1}{n+1}\right)^{k_n}\left(1 - \frac{2}{n+1}\right)^{k_n} - 1\right\}\right. \\
&\quad\quad \left. - 3\frac{n}{k_n}\left\{\left(1 - \frac{1}{n+1}\right)^{k_n} - 1\right\}\right]
\end{aligned} \tag{12}$$

for each $f \in \mathscr{C}^{(1)}[0, 1]$ with $f' \in \mathrm{Lip}_M 1$. Equation (12) will be used for the proof of the following theorem.

THEOREM 2. *Suppose* $f \in \mathscr{C}^{(2)}[0, 1]$ *and* $\lim_{n\to\infty}(k_n/n) = 0$. *Then*

$$\lim_{n\to\infty} \frac{n}{k_n} \{P_n^{k_n}(f; x) - f(x)\}$$
$$= \left(\frac{1}{2} x(1-x) f'(x)\right)' \qquad \textit{uniformly on } [0, 1].$$

Proof. Without loss of generality we may restrict ourselves on the subspace $\mathscr{C}_0^{(2)}[0, 1] = \{f \in \mathscr{C}^{(2)}[0, 1] \mid f(0) = 0 = f(1)\}$, which is a normed linear space equipped with the norm $q(f) = \|f''\|_\infty$. For $n \in \mathbb{N}_0$ we introduce operators $T_n: \mathscr{C}_0^{(2)}[0, 1] \to \mathscr{C}[0, 1]$ defining $T_0(f; x) = \frac{1}{2}x(1-x)f''(x)$ and (for $n \in \mathbb{N}$) $T_n(f; x) = (n/k_n)\{P_n^{k_n}(f; x) - f(x)\} - (n/k_n)\{(1 - 1/(n+1))^{k_n} - 1\}(x - \frac{1}{2})f'(x)$ $(x \in [0, 1])$. Regarding T_n as linear operators from $\mathscr{C}_0^{(2)}[0, 1]$ with the norm q into $\mathscr{C}[0, 1]$ with the supremum norm, we can estimate the associated operator norms. Since the right side of (12) is coefficientwise convergent to $(M/2)(x - x^2)$ as $n \to \infty$, there exists a constant $c > 0$ such that $|T_n(f; x)| \leqslant c$ for all $n \in \mathbb{N}$, $x \in [0, 1]$ provided $q(f) \leqslant 1$. Similarly, for T_0 there holds $|T_0(f; x)| \leqslant \frac{1}{8}$ for all $x \in [0, 1]$, provided $q(f) \leqslant 1$. Thus the norms of all operators T_n are uniformly bounded. For the polynomials in $\mathscr{C}_0^{(2)}[0, 1]$, there holds $\lim_{n\to\infty} T_n(f; x) = (\frac{1}{2}x(1-x)f'(x))' + (x - \frac{1}{2})f'(x) = T_0(f; x)$ uniformly on $[0, 1]$, on account of (7) and [8, Lemma 2]. Since the polynomials form a dense subspace in $\mathscr{C}_0^{(2)}[0, 1]$ we infer that $\lim_{n\to\infty} T_n(f; x) = T_0(f; x)$ uniformly on $[0, 1]$ for all $f \in \mathscr{C}_0^{(2)}[0, 1]$, which is equivalent with $\lim_{n\to\infty}(n/k_n)\{P_n^{k_n}(f; x) - f(x)\} = (\frac{1}{2}x(1-x)f'(x))'$ uniformly on $[0, 1]$ for all $f \in \mathscr{C}_0^{(2)}[0, 1]$. ∎

In the second case $\lim_{n\to\infty}(k_n/n) = \infty$, we cannot use the estimates from [7, Sects. 6, 7], which are valid only for limit operators admitting a Korovkin type theorem with finite test set $\mathscr{T}_k$. So the main difficulty in the proof of the following theorem consists in finding an uniform bound for the norms of T_n.

THEOREM 3. *Suppose* $f \in \mathscr{C}^{(1)}[0, 1]$ *and* $\lim_{n\to\infty}(k_n/n) = \infty$. *Then*

$$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{-k_n} \left\{P_n^{k_n}(f; x) - \int_0^1 f(t)\, dt\right\}$$
$$= \left(x - \frac{1}{2}\right) \int_0^1 6t(1-t) f'(t)\, dt$$

uniformly on $[0, 1]$.

Proof. Without loss of generality, we may restrict ourselves on the subspace $\mathscr{C}_0^{(1)}[0, 1] = \{f \in \mathscr{C}^{(1)}[0, 1] \mid \int_0^1 f(t)\, dt = 0\}$, which is a normed linear space equipped with the norm $q(f) = \|f'\|_\infty$. Each P_n is a linear operator from $\mathscr{C}_0^{(1)}[0, 1]$ with norm q into itself, on account of (1). We again try to estimate the associated operator norm, which will be denoted by $q(P_n)$, too. Suppose $q(f) \leqslant 1$. Then

$$|P_n(f; x)'| = \left| n(n+1) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} \right.$$
$$\left. \times \left\{ F\left(\frac{k+2}{n+1}\right) - 2F\left(\frac{k+1}{n+1}\right) + F\left(\frac{k}{n+1}\right) \right\} \right|$$
$$\leqslant n(n+1) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} (n+1)^{-2}$$
$$\leqslant 1 - \frac{1}{n+1}$$

for each $x \in [0, 1]$, $n \in \mathbb{N}$, which implies $q(P_n f) \leqslant 1 - 1/(n+1)$. Since $P_n(e_1 - \frac{1}{2}e_0) = (1 - 1/(n+1))(e_1 - \frac{1}{2}e_0)$, we obtain $q(P_n) = 1 - 1/(n+1)$. Then for the operators T_n defined by $T_n = (1 - 1/(n+1))^{-k_n} P_n^{k_n}$ there holds $q(T_n) = 1$ $(n \in \mathbb{N})$. Similarly for the operator $T_0: \mathscr{C}_0^{(1)}[0, 1] \to \mathscr{C}_0^{(1)}[0, 1]$ with $T_0(f; x) = (x - \frac{1}{2}) \int_0^1 6t(1-t) f'(t)\, dt$ there holds $q(T_0) = 1$. Thus the norms of all operators T_n $(n \in \mathbb{N}_0)$ are uniformly bounded by 1. The polynomials in $\mathscr{C}_0^{(1)}[0, 1]$ form a dense subspace, due to the Weierstrass theorem, and for each polynomial in $\mathscr{C}_0^{(1)}[0, 1]$, (9) entails $\lim_{n\to\infty} T_n f = T_0 f$ with respect to the norm q. Hence we have $\lim_{n\to\infty} T_n f = T_0 f$ for all $f \in \mathscr{C}_0^{(1)}[0, 1]$ with respect to the norm q. Also with respect to the supremum norm, $\lim_{n\to\infty} T_n f = T_0 f$ holds true. This follows from the representation $f(x) = \int_0^x f'(t)\, dt - \int_0^1 \int_0^t f'(w)\, dw\, dt$, which enables the estimate $|f(x)| \leqslant (3/2)\, q(f)$. ∎

Since $\int_0^1 6t(1-t)\, dt = 1$, the integral $\int_0^1 6t(1-t) f'(t)\, dt$ is a weighted mean of the derivative f'. Observe that the limit in Theorem 3 also can be written in the form $6(x - \frac{1}{2})(\int_0^1 f(t)\, dt - 2 \int_0^1 \int_0^t f(w)\, dw\, dt)$, where f' appears no more. This leads us to the conjecture that it will be possible to weaken the smoothness condition $f \in \mathscr{C}^{(1)}[0, 1]$ in Theorem 3.

In the last years, the approximation by Kantorovič operators was investigated by several authors [1, 5, 6, 9]. All authors consider the approximation with respect to some $\mathscr{L}_p$-norm. For the Kantorovič operators, $\mathscr{L}_p$-approximation seems to be a more suitable setting than uniform approximation. So we suggest as a problem to search for $\mathscr{L}_p$-versions of our Theorems 2 and 3 with weaker smoothness conditions for the function f.

To conclude this paper, we make a remark concerning saturation. For a function $f \in \mathscr{C}^{(2)}[0, 1]$ in the case $\lim_{n\to\infty}(k_n/n) = 0$, $\lim_{n\to\infty}(n/k_n)\{P_n^{k_n}(f; x) - f(x)\} = 0$ entails that f is constant and $P_n^{k_n} f = f$ for all $n \in \mathbb{N}$. In contrast to this suppose $f \in \mathscr{C}^{(1)}[0, 1]$ and $\lim_{n\to\infty}(k_n/n) = \infty$. Then $\lim_{n\to\infty}(1 - 1/(n+1))^{-k_n} \{P_n^{k_n}(f; x) - \int_0^1 f(t)\,dt\} = 0$ entails only $\int_0^1 t(1-t) f'(t)\,dt = 0$, and the example of the function $f(x) = x^2 - x + \frac{1}{6}$ shows that $P_n^{k_n} f = (1 - 1/(n+1))^{k_n} (1 - 2/(n+1))^{k_n} f \neq \int_0^1 f(t)\,dt = 0$ for infinitely many n is still possible.

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